

What Is Claimed Is:

- 1 1. A method for using a computer system to solve a system of
2 nonlinear equations specified by a vector function, \mathbf{f} , wherein $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ represents
3 $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$, wherein \mathbf{x} is a vector $(x_1, x_2, x_3, \dots, x_n)$,
4 the method comprising:
5 receiving a representation of an interval vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$,
6 wherein for each dimension, i , the representation of X_i includes a first floating-
7 point number, a_i , representing the left endpoint of X_i , and a second floating-point
8 number, b_i , representing the right endpoint of X_i ;
9 for each nonlinear equation $f_i(\mathbf{x}) = g(x'_j) - h(\mathbf{x}) = 0$ in the system of
10 equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, symbolically manipulating $f_i(\mathbf{x}) = 0$ within the computer system
11 to solve for any invertible term, $g(x'_j)$, thereby producing a modified equation
12 $g(x'_j) = h(\mathbf{x})$, wherein $g(x'_j)$ can be analytically inverted to produce an inverse
13 function $g^{-1}(\mathbf{y})$;
14 substituting the interval vector \mathbf{X} into the modified equation to produce the
15 equation $g(X'_j) = h(\mathbf{X})$;
16 solving for $X'_j = g^{-1}(h(\mathbf{X}))$; and
17 intersecting X'_j with the vector element X_j to produce a new interval vector
18 \mathbf{X}^+ ;
19 wherein the new interval vector \mathbf{X}^+ contains all solutions of the system of
20 equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ within the interval vector \mathbf{X} , and wherein the width of the new
21 interval vector \mathbf{X}^+ is less than or equal to the width of the interval vector \mathbf{X} .

- 1 2. The method of claim 1, further comprised of performing an
2 interval Newton step on \mathbf{X} to produce a resulting interval vector, \mathbf{Y} , wherein the
3 point of expansion of the interval Newton step is a point, \mathbf{x} , within the interval

4 vector \mathbf{X} , and wherein performing the interval Newton step involves evaluating
5 $\mathbf{f}(\mathbf{x})$ using interval arithmetic to produce an interval result $\mathbf{f}^I(\mathbf{x})$.

1 3. The method of claim 2, further comprising:
2 evaluating a first termination condition, wherein the first termination
3 condition is TRUE if,
4 zero is contained within $\mathbf{f}^I(\mathbf{x})$,
5 $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is regular, wherein $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is the Jacobian of the
6 function \mathbf{f} evaluated as a function of \mathbf{x} over the interval vector \mathbf{X} ,
7 and
8 \mathbf{Y} contained within \mathbf{X} ; and
9 if the first termination condition is TRUE, terminating and recording
10 $\mathbf{X} = \mathbf{X} \cap \mathbf{Y}$ as a final bound.

1 4. The method of claim 3, further comprising determining if $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is
2 regular by computing a pre-conditioned Jacobian, $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$, wherein \mathbf{B}
3 is an approximate inverse of the center of $\mathbf{J}(\mathbf{x}, \mathbf{X})$, and then solving for the interval
4 vector \mathbf{Y} that contains the value of \mathbf{y} that satisfies $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$, where $\mathbf{r}(\mathbf{x})$
5 $= -\mathbf{B}\mathbf{f}(\mathbf{x})$.

1 5. The method of claim 4, further comprising applying term
2 consistency to $\mathbf{B}\mathbf{f}(\mathbf{x}) = 0$.

1 6. The method of claim 1, wherein if no termination condition is
2 satisfied, the method further comprises returning to perform an interval Newton
3 step on the interval vector \mathbf{Y} .

1 7. The method of claim 6, wherein returning to perform the interval
2 Newton step on the interval vector \mathbf{Y} can involve splitting the interval vector
3 $\mathbf{X} = \mathbf{Y} \cap \mathbf{X}$.

1 8. The method of claim 2, further comprising:
2 evaluating a second termination condition;
3 wherein the second termination condition is TRUE if a function of the
4 width of the interval vector \mathbf{X} is less than a pre-specified value, ϵ_X , and the
5 absolute value of the function, \mathbf{f} , over the interval vector \mathbf{X} is less than a pre-
6 specified value, ϵ_F ; and
7 if the second termination condition is TRUE, terminating and recording \mathbf{X}
8 as a final bound.

1 9. The method of claim 1, wherein for each term, $g(x_j)$, that can be
2 analytically inverted within the equation $f_i(\mathbf{x}) = 0$, the method further comprises:
3 setting $X_j = X_j^+$ in \mathbf{X} ; and
4 repeating the process of symbolically manipulating, substituting, solving
5 and intersecting to produce the new interval vector X_j^+ .

1 10. The method of claim 1, wherein symbolically manipulating $f_i(\mathbf{x}) = 0$
2 involves selecting the invertible term $g(x_j)$ as the dominating term of the function
3 $f_i(\mathbf{x}) = 0$ within the interval vector \mathbf{X} .

1 11. A computer-readable storage medium storing instructions that
2 when executed by a computer cause the computer to perform a method for using a
3 computer system to solve a system of nonlinear equations specified by a vector

4 function, \mathbf{f} , wherein $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ represents $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$,
5 wherein \mathbf{x} is a vector $(x_1, x_2, x_3, \dots, x_n)$, the method comprising:
6 receiving a representation of an interval vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$,
7 wherein for each dimension, i , the representation of X_i includes a first floating-
8 point number, a_i , representing the left endpoint of X_i , and a second floating-point
9 number, b_i , representing the right endpoint of X_i ;
10 for each nonlinear equation $f_i(\mathbf{x}) = g(x'_i) - h(\mathbf{x}) = 0$ in the system of
11 equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, symbolically manipulating $f_i(\mathbf{x}) = 0$ within the computer system
12 to solve for any invertible term, $g(x'_i)$, thereby producing a modified equation
13 $g(x'_i) = h(\mathbf{x})$, wherein $g(x'_i)$ can be analytically inverted to produce an inverse
14 function $g^{-1}(y)$;
15 substituting the interval vector \mathbf{X} into the modified equation to produce the
16 equation $g(X'_i) = h(\mathbf{X})$;
17 solving for $X'_i = g^{-1}(h(\mathbf{X}))$; and
18 intersecting X'_i with the vector element X_i to produce a new interval vector
19 \mathbf{X}^+ ;
20 wherein the new interval vector \mathbf{X}^+ contains all solutions of the system of
21 equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ within the interval vector \mathbf{X} , and wherein the width of the new
22 interval vector \mathbf{X}^+ is less than or equal to the width of the interval vector \mathbf{X} .

1 12. The computer-readable storage medium of claim 11, wherein the
2 method further comprises performing an interval Newton step on \mathbf{X} to produce a
3 resulting interval vector, \mathbf{Y} , wherein the point of expansion of the interval Newton
4 step is a point, \mathbf{x} , within the interval vector \mathbf{X} , and wherein performing the interval
5 Newton step involves evaluating $\mathbf{f}(\mathbf{x})$ using interval arithmetic to produce an
6 interval result $\mathbf{f}^1(\mathbf{x})$.

1 13. The computer-readable storage medium of claim 12, wherein the
 2 method further comprises:
 3 evaluating a first termination condition, wherein the first termination
 4 condition is TRUE if,
 5 zero is contained within $\mathbf{f}^l(\mathbf{x})$,
 6 $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is regular, wherein $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is the Jacobian of the
 7 function \mathbf{f} evaluated as a function of \mathbf{x} over the interval vector \mathbf{X} ,
 8 and
 9 \mathbf{Y} is contained within \mathbf{X} ; and
 10 if the first termination condition is TRUE, terminating and recording
 11 $\mathbf{X} = \mathbf{X} \cap \mathbf{Y}$ as a final bound.

1 14. The computer-readable storage medium of claim 13, wherein the
 2 method further comprises determining if $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is regular by computing a pre-
 3 conditioned Jacobian, $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$, wherein \mathbf{B} is an approximate inverse of
 4 the center of $\mathbf{J}(\mathbf{x}, \mathbf{X})$, and then solving for the interval vector \mathbf{Y} that contains the
 5 value of \mathbf{y} that satisfies $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$, where $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$.

1 15. The computer-readable storage medium of claim 14, wherein the
 2 method further comprises applying term consistency to $\mathbf{B}\mathbf{f}(\mathbf{x}) = 0$.

1 16. The computer-readable storage medium of claim 11, wherein if no
 2 termination condition is satisfied, the method further comprises returning to
 3 perform an interval Newton step on the interval vector \mathbf{Y} .

1 17. The computer-readable storage medium of claim 16, wherein
2 returning to perform the interval Newton step on the interval vector \mathbf{Y} can involve
3 splitting the interval vector $\mathbf{X}=\mathbf{Y} \cap \mathbf{X}$.

1 18. The computer-readable storage medium of claim 12, wherein the
2 method further comprises:
3 evaluating a second termination condition;
4 wherein the second termination condition is TRUE if a function of the
5 width of the interval vector \mathbf{X} is less than a pre-specified value, ε_X , and the
6 absolute value of the function, \mathbf{f} , over the interval vector \mathbf{X} is less than a pre-
7 specified value, ε_F ; and
8 if the second termination condition is TRUE, terminating and recording \mathbf{X}
9 as a final bound.

1 19. The computer-readable storage medium of claim 11, wherein for
2 each term, $g(x_j)$, that can be analytically inverted within the equation $f_i(\mathbf{x}) = 0$, the
3 method further comprises:
4 setting $X_j = X_j^+$ in \mathbf{X} ; and
5 repeating the process of symbolically manipulating, substituting, solving
6 and intersecting to produce the new interval vector X_j^+ .

1 20. The computer-readable storage medium of claim 11, wherein
2 symbolically manipulating $f_i(\mathbf{x})=0$ involves selecting the invertible term $g(x_j)$ as
3 the dominating term of the function $f_i(\mathbf{x}) = 0$ within the interval vector \mathbf{X} .

1 21. An apparatus that uses a computer system to solve a system of
2 nonlinear equations specified by a vector function, \mathbf{f} , wherein $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ represents

3 $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$, wherein \mathbf{x} is a vector $(x_1, x_2, x_3, \dots, x_n)$,
4 the apparatus comprising:

5 a receiving mechanism that is configured to receive a representation of an
6 interval vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$, wherein for each dimension, i , the
7 representation of X_i includes a first floating-point number, a_i , representing the left
8 endpoint of X_i , and a second floating-point number, b_i , representing the right
9 endpoint of X_i ;

10 a symbolic manipulation mechanism, wherein for each nonlinear equation
11 $f_i(\mathbf{x}) = g(x'_j) - h(\mathbf{x}) = 0$ in the system of equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, the symbolic
12 manipulation mechanism is configured to manipulate $f_i(\mathbf{x}) = 0$ to solve for any
13 invertible term, $g(x'_j)$, thereby producing a modified equation $g(x'_j) = h(\mathbf{x})$,
14 wherein $g(x'_j)$ can be analytically inverted to produce an inverse function $g^{-1}(\mathbf{y})$;

15 a solving mechanism that is configured to,
16 substitute the interval vector \mathbf{X} into the modified equation
17 to produce the equation $g(X'_j) = h(\mathbf{X})$, and to
18 solve for $X'_j = g^{-1}(h(\mathbf{X}))$; and

19 an intersecting mechanism that is configured to intersect X'_j with the
20 vector element X_j to produce a new interval vector \mathbf{X}^+ , wherein the new interval
21 vector \mathbf{X}^+ contains all solutions of the system of equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ within the
22 interval vector \mathbf{X} , and wherein the width of the new interval vector \mathbf{X}^+ is less than
23 or equal to the width of the interval vector \mathbf{X} .

1 22. The apparatus of claim 21, further comprising an interval Newton
2 mechanism that is configured to perform an interval Newton step on \mathbf{X} to produce
3 a resulting interval vector, \mathbf{Y} , wherein the point of expansion of the interval
4 Newton step is a point, \mathbf{x} , within the interval vector \mathbf{X} , and wherein performing

5 the interval Newton step involves evaluating $\mathbf{f}(\mathbf{x})$ using interval arithmetic to
6 produce an interval result $\mathbf{f}^l(\mathbf{x})$.

1 23. The apparatus of claim 22, further comprising a termination
2 mechanism that is configured to:

3 evaluate a first termination condition, wherein the first termination
4 condition is TRUE if,

5 zero is contained within $\mathbf{f}^l(\mathbf{x})$,

6 $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is regular, wherein $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is the Jacobian of the
7 function \mathbf{f} evaluated as a function of \mathbf{x} over the interval vector \mathbf{X} ,

8 and

9 \mathbf{Y} is contained within \mathbf{X} ; and to

10 wherein if the first termination condition is TRUE, the termination
11 mechanism is configured to terminate and recording $\mathbf{X} = \mathbf{X} \cap \mathbf{Y}$ as a final bound.

1 24. The apparatus of claim 23, wherein the termination mechanism is
2 configured to determine if $\mathbf{J}(\mathbf{x}, \mathbf{X})$ is regular by computing a pre-conditioned
3 Jacobian; $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$, wherein \mathbf{B} is an approximate inverse of the center of
4 $\mathbf{J}(\mathbf{x}, \mathbf{X})$, and then to solve for the interval vector \mathbf{Y} that contains the value of \mathbf{y} that
5 satisfies $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$, where $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$.

1 25. The apparatus of claim 24, wherein the symbolic manipulation
2 mechanism is additionally configured to apply term consistency to $\mathbf{B}\mathbf{f}(\mathbf{x}) = 0$.

1 26. The apparatus of claim 21, wherein if no termination condition is
2 satisfied, the apparatus is configured to return to perform an interval Newton step
3 on the interval vector \mathbf{Y} .

1 27. The apparatus of claim 26, wherein returning to perform the
2 interval Newton step on the interval vector \mathbf{Y} can involve splitting the interval
3 vector $\mathbf{X} = \mathbf{Y} \cap \mathbf{X}$.

1 28. The apparatus of claim 22, wherein the termination mechanism
2 that is configured to:
3 evaluate a second termination condition;
4 wherein the second termination condition is TRUE if a function of the
5 width of the interval vector \mathbf{X} is less than a pre-specified value, ε_X , and the
6 absolute value of the function, \mathbf{f} , over the interval vector \mathbf{X} is less than a pre-
7 specified value, ε_F ; and
8 wherein if the second termination condition is TRUE, the termination
9 mechanism is configured to terminate and record \mathbf{X} as a final bound.

1 29. The apparatus of claim 21, wherein for each term, $g(x_j)$, that can be
2 analytically inverted within the equation $f_i(\mathbf{x}) = 0$, the apparatus is configured to:
3 set $X_j = X_j^+$ in \mathbf{X} ; and to
4 repeat the process of symbolically manipulating, substituting, solving and
5 intersecting to produce the new interval vector X_j^+ .

1 30. The apparatus of claim 21, wherein symbolically manipulating
2 $f_i(\mathbf{x}) = 0$ involves selecting the invertible term $g(x_j)$ as the dominating term of the
3 function $f_i(\mathbf{x}) = 0$ within the interval vector \mathbf{X} .